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RESEARCH ARTICLE

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Load sharing in series configuration

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Abstract

In reliability engineering, load sharing is typically associated with a system in *parallel* configuration. Examples include bridge support structures, electric power supply systems, and multiprocessor computing systems. We consider a reliability maximization problem for a high-voltage commutation device, wherein the total voltage across the device is shared by the components in *series* configuration. Here, the increase of the number of load-sharing components *increases component–level reliability* (as the voltage load per component reduces) but may *decrease system–level reliability* (because of the increased number of components in series). We provide the solution for the 2 popular life-load models: the proportional hazard and the accelerated failure time models with the underlying exponential and Weibull distributions for both a single and dual failure modes.

KEYWORDS

load-sharing systems, reliability optimization, system reliability

1 | INTRODUCTION

A load-sharing system is typically associated with a system in *parallel* configuration. In their renowned text on System Reliability Theory, Rausand and Hoyland[1] write, "Consider a parallel system with two identical components. The components share a common load." Another famous text by Kapur and Lamberson[2] also treats load-sharing systems *exclusively* as parallel systems.

Examples of parallel load-sharing systems include but are not limited to civil engineering (eg, structures[3]), materials engineering (eg, composites with fiber bundles[4]), power engineering (eg, distributed generation systems[5]), and computer/network engineering (multiprocessor computing systems[6]). Load-sharing systems are also often discussed in the context of (still parallel) systems with k-out-of-n configuration, eg Huang and Xu[7] and Amari and Bergman.[8]

Krivtsov and Gurevich[9] considered a reliability maximization problem for a high-voltage commutation device, wherein the total voltage across the device is shared by the components in *series* configuration. In this case, the increase of the number of load-sharing components increases component-level reliability (as the voltage load per component reduces) but may decrease system-level reliability (because of the increased number of components in series). Krivtsov & Gurevich derived the optimal number of the load-sharing components for the underlying exponential life distribution and the Arrhenius life-stress relationship.

In this paper, we generalize the solution to the 2 popular life-load models: the *proportional hazard* and the *accelerated failure time* with the underlying exponential and Weibull distributions. For these models, we provide some closed form solutions to optimal number of components that maximizes the system-level reliability. With that, we consider system's reliability as the objective function, but cost might be considered as well.

2 | LOAD-SHARING SYSTEMS IN SERIES CONFIGURATION

In many applications of electrical and power engineering, namely, in high-voltage power equipment, one often runs into problem of switching high voltages by solid-state devices, where the system voltage (10-100 kV) well exceeds the operating voltages of individual switching components (1-3 kV) such as silicon controlled rectifier (SCR), metal–oxide–semiconductor field-effect transistor (MOSFET), and Insulated-gate bipolar transistor (IGBT). One of the commonly practiced ways to increase the voltage capability of high-voltage switching devices is to put single switching components in a series configuration. Shown below is a high-voltage thyristor switch,[10] wherein the total switching voltage is shared by the serially connected thyristors.

In this case, the increase of the number of thyristors increases thyristor-level reliability (as the voltage load per thyristor reduces) but may decrease system-level reliability (because of the increased number of components in series). Clearly, the system reliability function in this case should have a maximum associated with an optimal number of thyristors in series.

Further, many engineering systems and their components can fail in dual failure modes. Examples include open and short for electrical and electronic devices, open and close for mechanical devices, stuck-at-0 and stuck-at-1 failures in combinatorial circuits, and fail safe and fail dangerous for safety monitoring systems.[11-13] Some typical examples of such components are an electronic diode, a switch, a relay, a sensor, a fluid flow valve, etc.

In general, electrical components comprising a load-sharing system as in Figure 1 are subject to 2 main failure modes: a *short circuit* of the semiconductor structure and an *open circuit* within the semiconductor structure. The odds of a short circuit vs an open circuit for solid state devices as diodes, thyristors, and transistors are typically distributed as 90% to 10%.[14] Let us consider the implications of these 2 failure modes to the system's reliability.

If one assumes some hot redundancy (in the form of additional serially connected components), then with a short-circuit failure mode, a series load-sharing system will logically be equivalent to a parallel one. This is because a component's short circuit would lead to the increase of the load for the remaining components, but the system would still be operational. From a *practical* standpoint, however, the failure of even one component (in such a complex and expensive system as a high-voltage thyristor switch) does not remain unnoticed, and the system is usually taken out of operation for a replacement of the failed component. The hot redundancy components in this case are intended to *temporarily* share the voltage load (to avoid the irreversible damage of the entire system) and are not supposed to operate long term. An important implication, therefore, is that a single component's short circuit would lead to the loss of system's functionality, as it needs to be taken out of operation for a repair. From this standpoint, a short circuit is essentially the same as an open circuit, as in both cases, the system loses its functionality.

Now let us consider the open-circuit failure mode. With the switch transitioning from the closed (nonconductive state) to the open (conductive state) state, initially, under a steep increase of the current (large di/dt), its high density over a small area of the semiconducting structure may result into a localized burn trough of this structure, thus leading to a short circuit within the component. Subsequently, under higher magnitudes of the current, the semiconducting structure may burn out altogether, thus resulting into an open circuit within the component. In this case, the entire system's load gets applied to this one component, which would inevitably cause its breakdown and yet another short circuit. The accompanying high-power arc would either (1) melt the metal casing and establish a conducting bridge or (2) would evaporate any metal elements and create a dielectric gap, capable of withstanding the entire system's voltage, thus creating a permanent open circuit. An important aspect for the open-circuit failure mode is that its likelihood does not really depend on component's voltage load but rather on the initial manufacturing quality of the semiconductor structure.

Another failure mode of a load-sharing electrical circuit in series configuration is related to their interconnections. As a result of cyclical thermal and mechanical stresses, the quality of a connection may eventually degrade and result into an open circuit. Again, the likelihood of this failure mode is practically independent of a

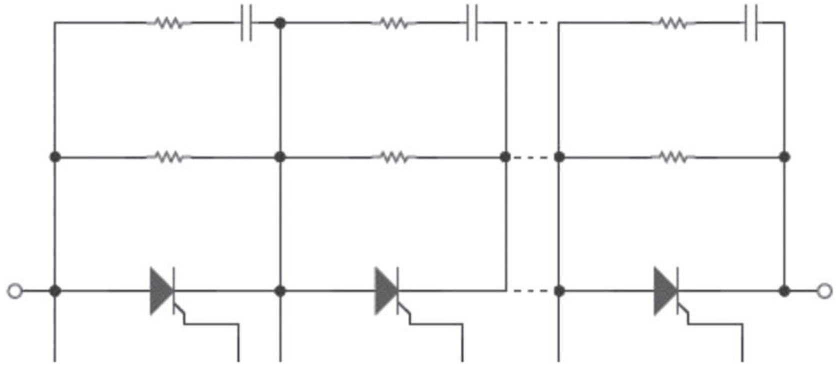

FIGURE 1 A serial load-sharing system, wherein total commutating voltage is shared by the serially connected SCRs (thyristors)

component's voltage load but may degrade system's reliability merely because of a number of connections in series.

Finally, there is a failure mode related to malfunctions of the switch's control system. For example, if an individual thyristor within the switch misses the open signal, then it remains closed while the rest are open. It means that the entire system voltage would be applied to this thyristor, which would cause its immediate breakdown (short circuit). Conversely, if an individual thyristor receives a false signal to close while the rest are open, it would cause the increased voltage for the remaining thyristors and, without hot redundancy, would lead to their breakdown and a short circuit of the entire switch.

On the basis of the above, we will first consider a short-circuit scenario (single failure mode), as it prevails in the series electrical load-sharing systems, and will then discuss a dual failure mode scenario allowing for both short- and open-circuit failures for each component under the competing risk model.

3 | RELIABILITY MATHEMATICS PRELIMINARIES

Let $h(t;L)$ be the load-dependent failure rate of a component. Two commonly used models relating failure time to a load level are the proportional hazard model (PHM) and the accelerated failure time model (AFTM).

3.1 | Proportional hazard model

Under the PHM, we have

$$h(t;\ L) = \psi(L)\times h(t;\ L_0) = \psi(L)\times h_0(t), \tag{1}$$

where $h_0(t)$ is the baseline failure rate and $\psi(L_0) = 1$. The commonly used model for $\psi(L)$ is the log-linear (exponential) law: $\psi(L) = \exp(\alpha_0 + \alpha_1 L)$. In fact, the exponential term can be replaced by any known positive, nondecreasing function.

For example, for the log-linear law,

$$\psi(L) = \exp(\alpha_0 + \alpha_1 L) = \delta\times\exp(\alpha_1 L). \tag{2}$$

For the power law,

$$\psi(L) = \delta\times \mathrm{L}^{\alpha}. \tag{3}$$

For the linear law,

$$\psi(L) = c + \alpha L. \tag{4}$$

The baseline failure rate can follow any time-varying function.

Further, for the cumulative hazard and the reliability functions, we have

$$H(t;L) = \psi(L)\times H_0(t), \tag{5}$$

$$\begin{aligned} R(t;L) = \exp\{-H(t;L)\} = \exp\{-\psi(L).H_0(t)\} \\ = [R_0(t)]^{\psi(L)}. \end{aligned} \tag{6}$$

3.2 | Accelerated failure time model

Under this model, the effect of the load is multiplicative in time. The reliability function is expressed as

$$R(t;L) = R_0(t\times\phi(L)), \tag{7}$$

where $R_0(.)$ is the reliability function at the baseline load. Function $\phi(L)$ represents the acceleration factor at load L. Without loss of generality, we can assume that $\phi(L_0) = 1$. When there is only one type of load, commonly used forms of $\phi(L)$ include the power law,

$$\phi(L) = \delta\times L^{\alpha}, \tag{8}$$

and the log–linear law,

$$\phi(L) = \delta\times\exp(\alpha L). \tag{9}$$

Finally, for the hazard functions, we have

$$H(t;L) = H_0(t\times\phi(L)), \tag{10}$$

$$h(t;L) = \phi(L)\times h_0(t\times\phi(L)). \tag{11}$$

It must be noted that if the baseline distribution is Weibull (or exponential) and the multiplicative factor (acceleration factor) follows the power law, then the AFTM and PHM coincide. However, in general, there is no direct duality between the models.

3.3 | System reliability under series load sharing

Consider a load-sharing series system with n components. The total load on the system is $L_T \equiv V$. The load is equally distributed between the components. Hence, the load on each component is

$$L \equiv \frac{L_T}{n} = \frac{V}{n}, \tag{12}$$

and the system reliability is

$$R_s(t;V,n) = [R(t;V/n)]^n. \tag{13}$$

Note that for fixed t, function $R\left(t;\frac{V}{n}\right)$ is an *increasing* function in n. However, function R^n is a *decreasing* function in n. Hence, there is an optimal value of n that maximizes the system reliability.

Alternatively, one can consider a logarithmic function of system reliability:

$$\ln R_s(t;V,n) = n\times\ln R(t;V/n). \tag{14}$$

The right-hand side of the above equation has 2 product terms. The first term increases with n, and the second term decreases with n. Further, it implies that

$$H_s(t;V,n) = n\times H\left(t;\frac{V}{n}\right). \tag{15}$$

To maximize the system reliability, we need to minimize the corresponding cumulative hazard function. Again, it has 2 product terms. The first term increases with the number of components, and the second term decreases with the number. Hence, there exist an optimal value that minimizes the cumulative hazard function and maximizes the system reliability.

4 | OPTIMAL NUMBER OF COMPONENTS IN A SERIES LOAD-SHARING SYSTEM FOR A SIGNLE FAILURE MODE

4.1 | Series load sharing under PHM

The system reliability in this case is

$$\begin{aligned} R_s(t;V,n) &= [R(t;L)]^n \equiv \left[R\left(t;\frac{V}{n}\right)\right]^n \\ &= \left[[R_0(t)]^{\psi(V/n)}\right]^n = [R_0(t)]^{n\times\psi(V/n)}. \end{aligned} \tag{16}$$

It follows that for fixed t, $R_0(t)$ is also fixed. Hence, for fixed t, to maximize the system reliability, we need to minimize $g(n)\equiv n\times\psi(V/n)$. Thus, the optimal n is independent of the form of the underlying failure time distribution $R_0(t)$ and is also independent of mission time t.

Under the log–linear law, we have

$$\psi(L) = \exp(\alpha_0+\alpha_1 L) = \delta\times\exp(\alpha_1 L). \tag{17}$$

For notational simplicity, hereafter, we will use α in the place of α_1. Hence,

$$\psi(L)\equiv\psi(V/n) = \delta\times\exp(\alpha\times V/n), \tag{18}$$

$$g(n)\equiv n\times\psi\left(\frac{V}{n}\right) = n\delta\times\exp(\alpha\times V/n). \tag{19}$$

The minimum of $g(n)$ can be obtained as

$$\frac{d}{dn}g(n)\equiv\delta\times\exp\left(\alpha\times\frac{V}{n}\right)+n\delta.\exp\left(\alpha\times\frac{V}{n}\right)\left(-\frac{\alpha V}{n^2}\right), \tag{20}$$

$$\frac{d}{dn}g(n) = 0\Rightarrow n = \alpha\times V. \tag{21}$$

Thus, the optimal n that maximizes the system reliability is $n = \|\alpha\times V\|$, where $\|.\|$ is the *nearest integer function*.

Figure 2 shows system-level reliability as a function of the number of the load-sharing components in series (with $\delta = 1$, $\alpha = 2$, and $V = 3$) for various values of component reliability. As expected, the optimal number does not depend on the latter.

It can be shown that for a PHM in the power law form, depending on model parameters, the optimal n will be either 1 or ∞. The linear law has a similar behavior. In both cases, the cost function can be used as regularization.

4.2 | Series load sharing under AFTM

The system reliability in this case is

$$\begin{aligned} R_s(t;V,n) &= [R(t;L)]^n \equiv [R_0(t\times\phi(L))]^n \\ &= [\exp\{-H_0(t\times\phi(L))\}]^n, \end{aligned} \tag{22}$$

$$\begin{aligned} R_s(t;V,n) &= \exp\{-n\times H_0(t.\phi(L))\} \\ &= \exp\{-H_s(t;V,n)\}, \end{aligned} \tag{23}$$

where

$$H_s(t;V,n) = n\times H_0(t\times\phi(L)). \tag{24}$$

From the equation above, it follows that (1) maximizing the system reliability is equivalent to minimizing the cumulative hazard function and (2) unlike the PHM, the optimal value does depend on the form of the underlying failure time distribution.

4.2.1 | AFTM with power law and the underlying exponential distribution

For the exponential distribution, we have

$$H_0(t) = \lambda t. \tag{25}$$

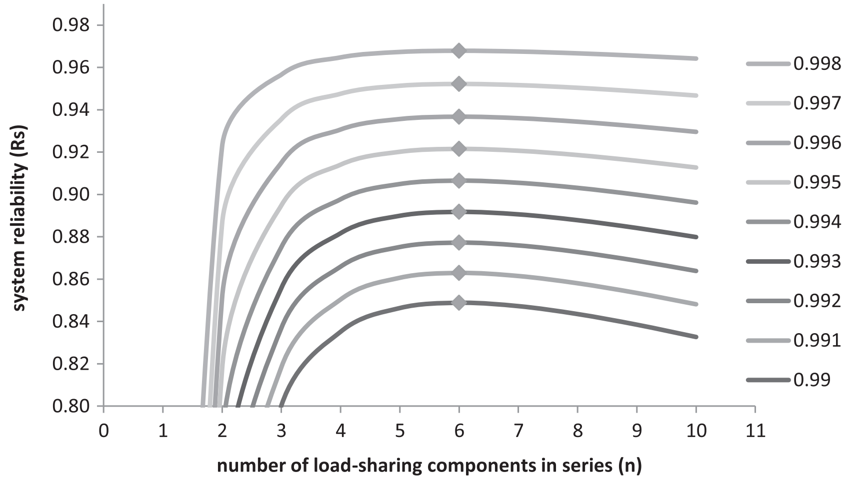

FIGURE 2 System-level reliability as a function of the number of the load-sharing components in series (with $\delta = 1$, $\alpha = 2$, and $V = 3$) for various values of component reliability. Optimal n is marked with the diamond

Hence,

$$H_s(t;V,n) = n\times\lambda t\times\phi(L) = n\times\lambda t\times\phi(V/n). \tag{26}$$

In this case, maximizing the system reliability is equivalent to minimizing $g(n)\equiv n\phi(V/n)$. The optimal n is independent of the mission time or even the baseline hazard function.

Now, for the power law,

$$\phi(L) = \delta\times L^{\alpha}, \tag{27}$$

and

$$g(n) = n\phi\left(\frac{V}{n}\right) = n\delta\left(\frac{V}{n}\right)^{\alpha}. \tag{28}$$

Similar to the PHM with the power law, depending on model parameters, the optimal n will again be either 1 or ∞. The cost function can be used to regularize this case.

4.2.2 | AFTM with log-linear law and the underlying exponential distribution

For the log-linear law,

$$\phi(L) = \delta\times\exp(\alpha L), \tag{29}$$

and

$$g(n) = n\phi\left(\frac{V}{n}\right) = n\delta\times\exp\left(\alpha\times\frac{V}{n}\right). \tag{30}$$

Note that the functional form of $g(n)$ is the same as in the PHM model with the log-linear law. Thus, the optimal n that maximizes the system reliability is

$$n = \|\alpha\times V\|. \tag{31}$$

4.2.3 | AFTM with power law and the underlying Weibull distribution

For the Weibull distribution, we have

$$H_0(t) = \left(\frac{t}{\eta}\right)^{\beta}. \tag{32}$$

Hence,

$$H_s(t;V,n) = n\times\left(\frac{t}{\eta}\phi(L)\right)^{\beta} = n\times\left(\frac{t}{\eta}\right)^{\beta}\times[\phi(V/n)]^{\beta}. \tag{33}$$

From the equation above, it follows that (1) maximizing the system reliability is equivalent to minimizing $g(n)\equiv n\times[\phi(V/n)]^{\beta}$ and (2) the optimal value of n is independent of the mission time and the scale parameter of the Weibull distribution.

For the power law,

$$\phi(L) = \delta\times L^{\alpha}, \tag{34}$$

and

$$g(n)\equiv n\left[\phi\left(\frac{V}{n}\right)\right]^{\beta} = n\left[\delta\times\left(\frac{V}{n}\right)^{\alpha}\right]^{\beta}. \tag{35}$$

Again, depending on model parameters, the optimal n will again be either 1 or ∞. The cost function can be used to regularize this case.

4.2.4 | AFTM with log-linear law and the underlying Weibull distribution

For the log-linear law,

$$\phi(L) = \delta\times\exp(\alpha L), \tag{36}$$

and

$$g(n) \equiv n\left[\phi\left(\frac{V}{n}\right)\right]^{\beta} = n\left[\delta \times \exp\left(\alpha \times \frac{V}{n}\right)\right]^{\beta}$$
$$= n \times \delta^{\beta} \times \exp\{\alpha\beta V/n\}. \quad (37)$$

Hence,

$$\frac{d}{dn}g(n) = 0 \Rightarrow n = \alpha \times \beta \times V. \quad (38)$$

Thus, the optimal n that maximizes the system reliability is

$$n = \|\alpha \times \beta \times V\|. \quad (39)$$

Figure 3 shows system-level reliability as a function of the number of the load-sharing components in series (with $\alpha = 2$ and $V = 3$) for various values of Weibull shape parameter with the scale parameter of $\eta = 1$ and mission time $t = 0.1$.

5 | OPTIMAL NUMBER OF COMPONENTS IN A SERIES LOAD-SHARING SYSTEM FOR DUAL FAILURE MODES

5.1 | Series load sharing under dual failure modes

The system reliability in this case is

$$R_s(t;\ V,\ n) = \left[R_{sh}\left(t;\ \frac{V}{n}\right)\right]^n \times \left[R_{op}(t)\right]^n, \quad (40)$$

where $R_{sh}(.)$ and $R_{op}(.)$ are the reliability functions relative to the short and open circuit, respectively.

Hence, the cumulative hazard function is

$$H_s(t; V, n) = \ n \times H_{sh}\left(t; \frac{V}{n}\right) + n \times H_{op}(t), \quad (41)$$

where $H_{sh}(.)$ and $H_{op}(.)$ are the cumulative hazard functions relative to the short and open circuit, respectively. Note that respective functions of the baseline distribution will be denoted as $H_{0,sh}(.)$ and $H_{0,op}(.) = H_{op}(.)$ because the probability of the open circuit is independent of the load.

5.2 | Dual failure modes under PHM

The system reliability in this case is

$$R_s(t; V, n) = \left[R_{sh}\left(t; \frac{V}{n}\right)\right]^n \times \left[R_{op}(t)\right]^n$$
$$= \left[\left[R_{0,sh}(t)\right]^{\psi(V/n)}\right]^n \times \left[R_{op}(t)\right]^n. \quad (42)$$

Let $\theta \equiv \theta(t) \equiv \ln(R_{op}(t))/\ln(R_{0,sh}(t))$. Hence, $R_{op}(t) = [R_{0,sh}(t)]^{\theta}$. Note that for a fixed mission time t, θ is constant. Hence,

$$R_s(t; V, n) = \left[R_{0,sh}(t)\right]^{n \times \psi(V/n) + n \times \theta}. \quad (43)$$

Hence, for fixed t, to maximize the system reliability, we need to minimize $g(n) \equiv n \times [\theta + \psi(V/n)]$. Thus, the optimal n is independent of the form of the underlying failure time distributions $R_{0,sh}(t)$ and $R_{op}(t)$. However, in this case, the optimal value is dependent on θ, which can vary with the mission time t.

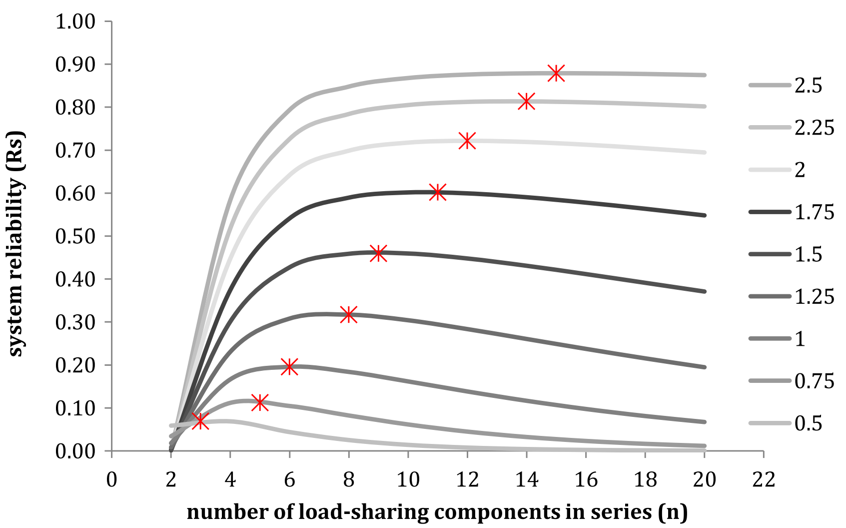

FIGURE 3 System-level reliability as a function of the number of the load-sharing components in series (with $\alpha = 2$ and $V = 3$) for various values of Weibull shape parameter with the scale parameter of $\eta = 1$ and mission time $t = 0.1$. Optimal n is marked with the asterisk [Colour figure can be viewed at wileyonlinelibrary.com]

5.2.1 | Dual failure modes under PHM with log-linear law

Under the log-linear law, we have $g(n) \equiv n \times [\theta + \delta \times \exp(\alpha \times V/n)]$.

The minimum of $g(n)$ can be obtained as the value of n that satisfies the following equation:

$$\frac{d}{dn}g(n) \equiv \theta + \delta \times \exp\left(\alpha \times \frac{V}{n}\right) + n\delta \times \exp\left(\alpha \times \frac{V}{n}\right)\left(-\frac{\alpha V}{n^2}\right) = 0. \quad (44)$$

Define $x \equiv \frac{\alpha V}{n}$. Hence, the solution must satisfy the following equation: $\theta + \delta \times (1 - x)e^x = 0$, or

$$(x-1)e^x = \theta/\delta. \quad (45)$$

Define $\kappa(x) \equiv (x-1)e^x$. Hence, the solution to x in the above equation is $k^{-1}(\theta/\delta)$.

a. The solution to the above equation can be found numerically (eg, using bisection, which has a logarithmic computational complexity with respect to solution accuracy). If the value of x corresponding to the solution is determined, then the optimal $n = \|\alpha \times V/x\|$.
b. Because $[R_{op}(t)]^n$ is a decreasing function in n, the optimal value of n that maximizes the system reliability should be less than $n = \alpha \times V$. Hence, the upper bound on the optimal value is $n = \|\alpha \times V\|$.
c. From the definitions, θ and δ are always positive; hence, x must be greater than 1. Further, θ is much smaller than 1. Hence, in most cases, x will be less than 2. In such cases, we can find an approximate solution to x as

$$x \approx \sqrt{1 + \theta/\delta},$$
$$n \approx \frac{\alpha \times V}{\sqrt{1 + \theta/\delta}}.$$

Table 1 and Figure 4 show system-level reliability as a function of the number of the load-sharing components in series (with = 1, = 2, and $V = 3$) for various values of θ and $R_{0,sh}(.) = 0.995$.

5.2.2 | Dual failure modes under PHM with power law

Under the log-linear law, we have

$$g(n) \equiv n \times [\theta + \delta \times \exp(\alpha \times V/n)]. \quad (46)$$

Under the power law, we have

$$g(n) \equiv n \times [\theta + \delta \times (V/n)^{\alpha}]. \quad (47)$$

The minimum of $g(n)$ can be obtained as the value of n that satisfied the following equation:

$$\frac{d}{dn}g(n) \equiv \theta + \delta \times \left(\frac{V}{n}\right)^{\alpha} + n\delta\alpha \times \left(\frac{V}{n}\right)^{\alpha-1}\left(-\frac{V}{n^2}\right) = \theta + \delta \times (1-\alpha)\left(\frac{V}{n}\right)^{\alpha}, \quad (48)$$

$$\frac{d}{dn}g(n) = 0 \Rightarrow n = V\left[\frac{\theta}{\delta(\alpha-1)}\right]^{-\frac{1}{\alpha}} \equiv \omega \times V. \quad (49)$$

Thus, the optimal n that maximizes the system reliability is

TABLE 1 System-level reliability as a function of the number of the load-sharing components in series (with $\delta = 1$, $\alpha = 2$, and $V = 3$) for various values of θ and $R_{0,sh}(.) = 0.995$

		$R_s(^*)$ under n=									
θ	x	1	2	3	4	5	6	7	8	9	10
0.1	1.036	0.132	0.817	0.893	0.912	0.918	0.919	0.917	0.915	0.912	0.908
0.2	1.069	0.132	0.816	0.892	0.910	0.916	0.916	0.914	0.911	0.908	0.904
0.3	1.100	0.132	0.815	0.891	0.909	0.913	0.913	0.911	0.908	0.904	0.899
0.4	1.129	0.132	0.814	0.889	0.907	0.911	0.910	0.908	0.904	0.900	0.895
0.5	1.157	0.132	0.814	0.888	0.905	0.909	0.908	0.905	0.900	0.895	0.890
0.6	1.184	0.132	0.813	0.887	0.903	0.906	0.905	0.901	0.897	0.891	0.886
0.7	1.209	0.132	0.812	0.885	0.901	0.904	0.902	0.898	0.893	0.887	0.881
0.8	1.233	0.132	0.811	0.884	0.900	0.902	0.900	0.895	0.890	0.883	0.877
0.9	1.256	0.132	0.810	0.883	0.898	0.900	0.897	0.892	0.886	0.879	0.872

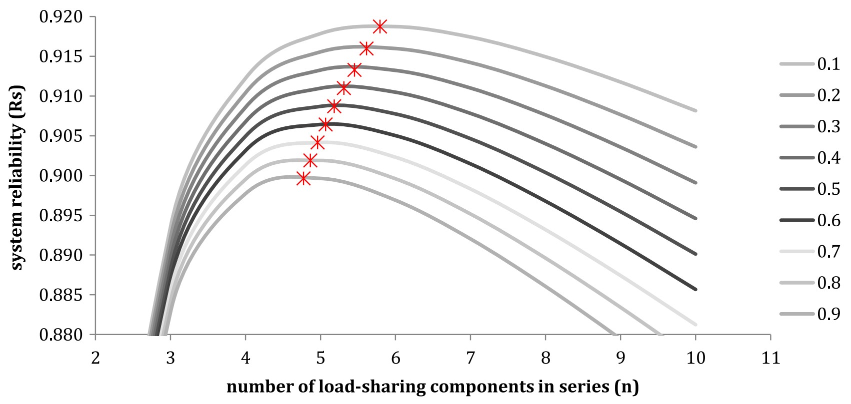

FIGURE 4 System-level reliability (with $\delta = 1$, $\alpha = 2$, and $V = 3$) for various values of θ and $R_{0,sh}(.) = 0.995$. Optimal n is marked with the asterisk [Colour figure can be viewed at wileyonlinelibrary.com]

$$n = \|\omega \times V\|, \text{where } \omega \equiv \left[\frac{\theta}{\delta(\alpha-1)}\right]^{-\frac{1}{\alpha}}. \tag{50}$$

Note that unlike single failure mode model, the dual failure model has an optimal value for n even when the underlying life-stress model follows the power law.

Figure 5 shows system-level reliability as a function of the number of the load-sharing components in series (with $\delta = 1$, $\alpha = 2$, and $V = 3$) for various values of θ and $R_{0,sh}(.) = 0.995$.

5.3 | Dual failure modes under AFTM

The system reliability in this case is

$$\begin{aligned} R_s(t;V,n) &= \left[R_{sh}\left(t;\frac{V}{n}\right)\right]^n \times \left[R_{op}(t)\right]^n \\ &= \left[R_{0,sh}(t \times \phi(V/n))\right]^n \times \left[R_{op}(t)\right]^n. \end{aligned} \tag{51}$$

Hence, the cumulative hazard function is

$$\begin{aligned} H_s(t;\ V,\ n) &= n \times H_{0,sh}(t \times \phi(V/n)) \\ &\quad + n \times H_{op}(t), \end{aligned} \tag{52}$$

where $H_{sh}(.)$ and $H_{op}(.)$ are the cumulative hazard functions relative to the short and open circuit, respectively.

5.3.1 | Dual failure mode AFTM with log-linear law and the underlying exponential distribution

For the log-linear law,

$$\begin{aligned} &\phi(L) = \delta \times \exp(\alpha L), \\ &g(n) \equiv n\left[\theta + \phi\left(\frac{V}{n}\right)\right] = n\theta + n\delta \times \exp\left(\alpha \times \frac{V}{n}\right). \end{aligned} \tag{53}$$

Note that the form of $g(n)$ in this case is similar to the case of dual failure modes subjected to log-linear form of PHM relationship. Hence, the same solution as in section 5.2.1 is applicable.

5.3.2 | Dual failure mode AFTM with power law and the underlying exponential distribution

For the exponential distribution, we have

$$H_{0,sh}(t) = \lambda_{sh} t, \tag{54}$$

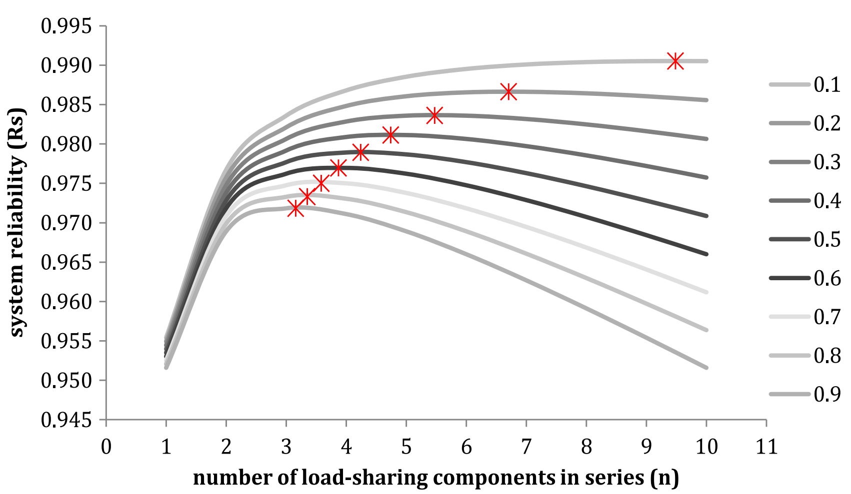

FIGURE 5 System-level reliability (with $\delta = 1$, $\alpha = 2$, and $V = 3$) for various values of θ and $R_{0,sh}(.) = 0.995$. Optimal n is marked with the asterisk [Colour figure can be viewed at wileyonlinelibrary.com]

$$H_{op}(t) = \lambda_{op} t. \tag{55}$$

Hence,

$$H_s(t; V, n) = n \times \lambda_{sh} \times t \times \phi\left(\frac{V}{n}\right) + n \times \lambda_{op} \times t. \tag{56}$$

In this case, maximizing the system reliability is equivalent to minimizing $g(n) \equiv n\left[\theta + \phi\left(\frac{V}{n}\right)\right]$, where $\theta = \frac{\lambda_{op}}{\lambda_{sh}}$. The optimal n is independent of the mission time but depends on the ratio of the baseline hazard functions for open- and short-failure modes.

Now, for the power law,

$$\phi(L) = \delta \times L^{\alpha}, \tag{57}$$

$$g(n) \equiv n \times [\theta + \delta \times (V/n)^{\alpha}]. \tag{58}$$

Note that the form of $g(n)$ in this case is similar to the case of dual failure modes subjected to power law form of PHM relationship. Hence, the same solution as in section 5.2.2 is applicable.

5.3.3 | Dual failure mode AFTM with log-linear law and the underlying Weibull distribution

For the log-linear law,

$$\phi(L) = \delta \times \exp(\alpha L), \tag{59}$$

and

$$g(n) \equiv n\left\{\theta + \left[\phi\left(\frac{V}{n}\right)\right]^{\beta}\right\} = n\left\{\theta + \left[\delta \times \exp\left(\alpha \times \frac{V}{n}\right)\right]^{\beta}\right\}. \tag{60}$$

Hence,

$$g(n) \equiv n \times \left[\theta + \delta^{\beta} \times \exp\{\alpha\beta V/n\}\right]. \tag{61}$$

Note that the form of $g(n)$ in this case is similar to the case of dual failure modes subjected to log-linear form of PHM relationship. The only difference is that α and δ should be replaced $\alpha\beta$ and δ^{β}, respectively. Hence, the same form of the solution is applicable for this case. Hence, the same solution as in section 5.2.1 is applicable.

5.3.4 | Dual failure mode AFTM with power law and the underlying Weibull distribution

For the Weibull distribution (with same shape parameter, β), we have

$$H_{0,sh}(t) = \left(\frac{t}{\eta_{sh}}\right)^{\beta}, \tag{62}$$

$$H_{op}(t) = \left(\frac{t}{\eta_{op}}\right)^{\beta}. \tag{63}$$

Hence,

$$H_s(t; V, n) = n \times \left(\frac{t \times \phi\left(\frac{V}{n}\right)}{\eta_{sh}}\right)^{\beta} + n \times \left(\frac{t}{\eta_{op}}\right)^{\beta}. \tag{64}$$

Let $\theta \equiv \left(\frac{\eta_{sh}}{\eta_{op}}\right)^{\beta}$. After rewriting the above equation, we have

$$H_s(t; V, n) = \left(\frac{t}{\eta_{sh}}\right)^{\beta} \times n \times \left[\left(\phi\left(\frac{V}{n}\right)\right)^{\beta} + \theta\right]. \tag{65}$$

From the equation above, it follows that (1) maximizing the system reliability is equivalent to minimizing $g(n) \equiv n \times \{\theta + [\phi(V/n)]^{\beta}\}$ and (2) the optimal value of n is independent of the mission time but depends on the ratio of the scale parameters for open- and short-circuit failure modes.

For the power law,

$$\phi(L) = \delta \times L^{\alpha}, \tag{66}$$

and

$$g(n) \equiv n\left\{\theta + \left[\phi\left(\frac{V}{n}\right)\right]^{\beta}\right\} = n\left\{\theta + \left[\delta \times \left(\frac{V}{n}\right)^{\alpha}\right]^{\beta}\right\} = n\left\{\theta + \delta^{\beta} \times \left(\frac{V}{n}\right)^{\alpha\beta}\right\}. \tag{67}$$

Note that the form of $g(n)$ in this case is similar to the case of dual failure modes subjected to power law form of PHM relationship. The only difference is that α and δ should be replaced $\alpha\beta$ and δ^{β}, respectively. Hence, the same form of the solution as in section 5.2.2 is applicable.

TABLE 2 Solutions for optimal number of components under series load sharing for various underlying distributions and life-load relationships

Underlying Distribution	Life-load Model	Single Failure Mode	Dual Failure Modes
Arbitrary	PHM with log-linear law	$n = \|\| \alpha \times V \|\|$	$n = \|\|\alpha \times V/x\|\|$ $x = \kappa^{-1}(\theta/\delta) \approx \sqrt{1+\theta/\delta}$ $\theta = \ln(R_{op}(t))/\ln(R_{0,sh}(t))$
	PHM with power law	$n = \arg\min\left\{n\delta\left(\frac{V}{n}\right)^{\alpha}\right\}$ Regularization required	$n = V\left[\frac{\theta}{\delta(\alpha-1)}\right]^{-\frac{1}{\alpha}} \equiv \|\|\omega \times V\|\|$ $\theta = \ln(R_{op}(t))/\ln(R_{0,sh}(t))$
Exponential	AFTM with log-linear law	$n = \|\|\alpha \times V\|\|$	$n = \|\|\alpha \times V/x\|\|$ $x = \kappa^{-1}(\theta/\delta) \approx \sqrt{1+\theta/\delta}$ $\theta = \lambda_{op}/\lambda_{sh}$
	AFTM with power law	$n = \arg\min\left\{n\delta\left(\frac{V}{n}\right)^{\alpha}\right\}$ Regularization required	$n = V\left[\frac{\theta}{\delta(\alpha-1)}\right]^{-\frac{1}{\alpha}} \equiv \|\|\omega \times V\|\|$ $\theta = \lambda_{op}/\lambda_{sh}$
Weibull	AFTM with log-linear law	$n = \|\|\alpha \times \beta \times V\|\|$	$n = \|\|\alpha \times \beta \times V/x\|\|$ $x \approx \sqrt{1+\theta/\delta^{\beta}}, \ \theta \equiv \left(\frac{\eta_{sh}}{\eta_{op}}\right)^{\beta}$
	AFTM with power law	$n = \arg\min\left\{n \times \delta^{\beta} \times \left(\frac{V}{n}\right)^{\alpha\beta}\right\}$ Regularization required	$n = V\left[\frac{\theta}{\delta^{\beta}(\alpha\beta-1)}\right]^{-\frac{1}{\alpha\beta}} \equiv \|\|\omega \times V\|\|$ $\theta \equiv \left(\frac{\eta_{sh}}{\eta_{op}}\right)^{\beta}$

Abbreviations: AFTM, accelerated failure time model; PHM, proportional hazard model.

6 | CONCLUDING REMARKS

Load sharing in series configuration poses a problem of selecting the optimal number of serially connected components that would maximize the system-level reliability. This is because the increase of the number of load-sharing components increases component-level reliability but may decrease system-level reliability.

The series load-sharing systems play an important role in the electrical engineering applications, wherein the 2 failure modes of interest are short and open circuit. These two have different implications to the system's reliability. The likelihood of the former is dependent on the voltage load while that of the latter is mostly independent of it. Both failure modes, however, lead to the loss of the system's functionality, although an open circuit causes immediate loss while a short circuit may cause eventual loss. The paper addressed both single and dual failure mode scenarios under two popular models for load-dependent failure rates: the PHM and the AFTM with the underlying exponential and Weibull distribution.

Table 2 summarizes solutions for optimal number of components under series load sharing for various underlying distributions and life-load relationships.

An important aspect of practical application of the proposed solution is component's *nominal voltage rating*. Consider the following scenario: The overall voltage that needs to be handled by the system is 100 V, and the optimal n turns out to be 6 or 16.(6) V/component. However, the closest available component's nominal voltage rating is 20 V, so one would have to put 5 (instead of 6) components in series, which, strictly speaking, is no longer the optimal number. While the derivation of optimal n under this constraint is the subject of a separate paper, one can observe from Figures 2–4, that the objective function in the neighborhood of the extremum is relatively flat. This allows for adjacent values around the optimal n to accommodate component's nominal voltage rating without a significant loss in system's reliability.

Proposed methodology can be extended to arbitrary failure time distribution and arbitrary load-life relationships. While the main application of this method lies within *electrical engineering* (such as serially connected thyristors/diodes, or resistors in voltage dividers, or series contacts of a high-voltage switching system[15]), it can still be extrapolated to other engineering fields with a similar load-sharing context. The proposed model can also be extended to the cases of uncovered and/or propagated failures where certain failure mode of a component can propagate to other parts of the system and eventually leads to the entire system failure.[16]

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